

QUENCHED LARGE DEVIATIONS FOR MULTIDIMENSIONAL RANDOM WALK IN RANDOM ENVIRONMENT WITH HOLDING TIMES

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ABSTRACT. We consider a random walk in random environment with random holding times, that is, the random walk jumping to one of its nearest neighbors with some transition probability after a random holding time. Both the transition probabilities and the laws of the holding times are randomly distributed over the integer lattice. Our main result is quenched large deviation principles for the position of the random walk. The rate function is given by the Legendre transform of so-called Lyapunov exponents for the Laplace transform of the first passage time. By using this representation, we derive some asymptotics of the rate function in some special cases.

1. INTRODUCTION

In this paper, we study large deviations for random walk in random environment with random holding times. The same problem has been studied by Dembo, Gantert, and Zeitouni [3] in one-dimensional case. They assumed that the transition probabilities are uniform elliptic and holding times bounded away from zero but otherwise only quite general ergodicity and integrability conditions. We consider the multidimensional case with rather restrictive independence assumptions: the transition probability and holding times are i.i.d. and mutually independent. On the other hand, we need a weaker ellipticity assumption and also do not assume holding times bounded below.

We now describe the setting in more detail. Denote by \mathcal{P}_1 the space of probability measures on the set $\{e \in \mathbb{Z}^d; |e| = 1\}$ of the canonical unit vectors of \mathbb{R}^d . Let $\Omega := \mathcal{P}_1^{\mathbb{Z}^d}$ be the space equipped with the canonical product σ -field \mathcal{G} and an i.i.d. probability measure \mathbb{P} . Then, for an *environment* $\omega = (\omega(x, \cdot))_{x \in \mathbb{Z}^d} \in \Omega$ and $z \in \mathbb{Z}^d$, the *random walk in random environment* (RWRE for short) is the Markov chain $(X = (X_n)_{n=0}^\infty, (P_\omega^x)_{x \in \mathbb{Z}^d})$ on \mathbb{Z}^d defined as follows: $P_\omega^x(X_0 = x) = 1$ and

$$P_\omega^z(X_{n+1} = x + e | X_n = x) = \omega(x, e)$$

for all $x \in \mathbb{Z}^d$, $n \in \mathbb{N}_0$ and $e \in \mathbb{Z}^d$ with $|e| = 1$.

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Let \mathcal{P}_2 be the space of Borel probability measures on $(0, \infty)$. We consider the space $\Sigma := \mathcal{P}_2^{\mathbb{Z}^d}$ endowed with the canonical σ -field \mathcal{S} and an i.i.d. probability measure \mathbf{P} . Denote an element of Σ by $\sigma = (\sigma_x)_{x \in \mathbb{Z}^d}$, and let $\tau = (\tau_n(x))_{n \in \mathbb{N}_0, x \in \mathbb{Z}^d} \in (0, \infty)^{\mathbb{N}_0 \times \mathbb{Z}^d}$ be independent random variables with σ_x being the law of $\tau_n(x)$ for each $n \in \mathbb{N}_0$. We call $(\tau_n(x))_{n \in \mathbb{N}_0, x \in \mathbb{Z}^d}$ *holding times* and denote by P_σ^{HT} their law, that is, $\sigma_x(ds) = P_\sigma^{\text{HT}}(\tau_1(x) \in ds)$.

For a random walk path X and holding times τ , we define the corresponding continuous-time random walk path $(Z_t)_{t \geq 0}$ as follows:

$$Z_t := \begin{cases} X_n, & \sum_{m=0}^{n-1} \tau_m(X_m) \leq t < \sum_{m=0}^n \tau_m(X_m), \\ \Delta, & t \geq \sum_{m=0}^\infty \tau_m(X_m), \end{cases}$$

where $\sum_{m=0}^{n-1} \tau_m(X_m) := 0$ if $n = 0$ and Δ is the graveyard for $(Z_t)_{t \geq 0}$. This process $(Z_t)_{t \geq 0}$ is called *random walk in random environment with holding times (RWREHT)* for short). Let $\tilde{P}_{\omega, \sigma}^x := P_\omega^x \otimes P_\sigma^{\text{HT}}$ and denote the expectations with respect to \mathbb{P} , P_ω^x , \mathbf{P} , P_σ^{HT} and $\tilde{P}_{\omega, \sigma}^x$ by \mathbb{E} , E_ω^x , \mathbf{E} , E_σ^{HT} and $\tilde{E}_{\omega, \sigma}^x$, respectively. Throughout this paper, we make the following assumptions.

Assumption 1.1 (1) $\log \min_{|e|=1} \omega(0, e) \in L^d(\mathbb{P})$ and $\int_0^\infty s \sigma_0(ds) \in L^d(\mathbf{P})$,
 (2) $0 \in \text{conv}\left(\text{supp}\left(\text{law}\left(\sum_{|e|=1} \omega(0, e)e\right)\right)\right)$.

By the first assumption and Jensen's inequality, we have

$$\theta_{\lambda, \sigma}(z) := -\log \int_0^\infty e^{-\lambda s} \sigma_z(ds) \in L^d(\mathbf{P})$$

for each $\lambda > 0$ and $z \in \mathbb{Z}^d$. The second assumption is called *nestling property* and will be used only in the proof of the large deviation lower bound.

We prove a large deviation principle for the law of scaled position Z_t/t of *RWREHT* following the same strategy as in [9]. We introduce $H^Z(y) := \inf\{t \geq 0; Z_t = y\}$ as the first passage time through y for the path $(Z_t)_{t \geq 0}$ and study the asymptotics of the cumulant generating function as $y \rightarrow \infty$ first. Define for any $\lambda \geq 0$, $\omega \in \Omega$, $\sigma \in \Sigma$ and $x, y \in \mathbb{Z}^d$,

$$\begin{aligned} e_\lambda(x, y, \omega, \sigma) &:= \tilde{E}_{\omega, \sigma}^x[\exp\{-\lambda H^Z(y)\} \mathbb{1}_{\{H^Z(y) < \infty\}}], \\ a_\lambda(x, y, \omega, \sigma) &:= -\log e_\lambda(x, y, \omega, \sigma), \\ d_\lambda(x, y, \omega, \sigma) &:= \max\{a_\lambda(x, y, \omega, \sigma), a_\lambda(y, x, \omega, \sigma)\}. \end{aligned}$$

Theorem 1.2 For each $\lambda \geq 0$, there exists a nonrandom function $\alpha_\lambda : \mathbb{R}^d \rightarrow [0, \infty)$ such that for all $x \in \mathbb{Z}^d$,

$$\begin{aligned} (1.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} a_\lambda(0, nx, \omega, \sigma) &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \otimes \mathbf{E}[a_\lambda(0, nx, \omega, \sigma)] \\ &= \inf_{n \geq 1} \frac{1}{n} \mathbb{E} \otimes \mathbf{E}[a_\lambda(0, nx, \omega, \sigma)] = \alpha_\lambda(x) \end{aligned}$$

holds $\mathbb{P} \otimes \mathbf{P}$ -a.s. and in $L^1(\mathbb{P} \otimes \mathbf{P})$. Moreover α_λ has the following properties: for any $q > 0$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}\alpha_\lambda(qx) &= q\alpha_\lambda(x), \\ \alpha_\lambda(x+y) &\leq \alpha_\lambda(x) + \alpha_\lambda(y),\end{aligned}$$

and

$$|x|(-\log \mathbf{E}[\exp\{-\theta_{\lambda,\sigma}(0)\}]) \leq \alpha_\lambda(x) \leq |x|\left(\max_{|e|=1} \mathbf{E}[-\log \omega(0, e)] + \mathbf{E}[\theta_{\lambda,\sigma}(0)]\right).$$

Furthermore, $\alpha_\lambda(x)$ is concave increasing in $\lambda \geq 0$ and convex in $x \in \mathbb{R}^d$. In particular, it is jointly continuous in $\lambda \geq 0$ and $x \in \mathbb{R}^d$.

Theorem 1.3 The following holds $\mathbb{P} \otimes \mathbf{P}$ -a.s. and in $L^1(\mathbb{P} \otimes \mathbf{P})$: for all $\lambda \geq 0$ and all sequences $(x_n)_{n=1}^\infty$ of \mathbb{R}^d with $|x_n| \rightarrow \infty$,

$$(1.2) \quad \lim_{n \rightarrow \infty} \frac{a_\lambda(0, [x_n], \omega, \sigma) - \alpha_\lambda(x_n)}{|x_n|} = 0,$$

where $[x_n]$ denotes a point in \mathbb{Z}^d that is closest to x_n in l^1 -distance.

The following theorem is our main result.

Theorem 1.4 The law of Z_t/t obeys the following large deviation principle with rate function

$$I(x) := \sup_{\lambda \geq 0} (\alpha_\lambda(x) - \lambda) :$$

- Upper bound: for any closed subset $A \subset \mathbb{R}^d$, we have $\mathbb{P} \otimes \mathbf{P}$ -a.s.,

$$(1.3) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \log \tilde{P}_{\omega, \sigma}^0(Z_t \in tA) \leq - \inf_{x \in A} I(x).$$

- Lower bound: for any open subset $B \subset \mathbb{R}^d$, we have $\mathbb{P} \otimes \mathbf{P}$ -a.s.,

$$(1.4) \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \log \tilde{P}_{\omega, \sigma}^0(Z_t \in tB) \geq - \inf_{x \in B} I(x).$$

1.1. Comments on the proof. We basically follow the strategy taken in [9, 10]. The second reference [10] is the first paper studying large deviations for multidimensional RWRE, where nestling walks in i.i.d. random environment are considered. After that, several generalizations were discussed by different methods: Varadhan [8] generalized Zerner's result to general ergodic environments and also obtained large deviations under the so-called *annealed* measure; Rassoul-Agha and Seppäläinen [6] obtained process level large deviations. Although our method requires rather restrictive independence assumptions and only proves level-1 large deviation principle, it has an advantage of giving a relatively simple representation of the rate function. This for instance allows us to determine the asymptotics of the rate function as $x \rightarrow \infty$ and $x \rightarrow 0$ in some special cases, see Section 6.

Next, we explain the outline of our proof of the large deviation principle. To prove a large deviation for random walk in random environment, it is standard to consider the Laplace transform of $H^Z(y)$. Indeed, the large deviation upper bound is almost immediate from Theorem 1.3 since for a compact set $K \subset \mathbb{R}^d$,

$$\begin{aligned} \tilde{P}_{\omega,\sigma}^0(Z_t \in tK) &\leq \#(tK \cap \mathbb{Z}^d) \max_{y \in tK \cap \mathbb{Z}^d} \tilde{P}_{\omega,\sigma}^0(H^Z(y) \leq t) \\ &\leq \#(tK \cap \mathbb{Z}^d) \max_{y \in tK \cap \mathbb{Z}^d} e^{-\lambda t} e_\lambda(0, y, \omega, \sigma) \\ &= \max_{y \in tK \cap \mathbb{Z}^d} \exp\{-t(\alpha_\lambda(y/t) - \lambda) + o(t)\}. \end{aligned}$$

In order to extend this to general closed sets, we have to check the so-called exponential tightness but it is not hard (see page 10). The proof of Theorem 1.3 itself is based on the fact that our e_λ is the survival probability for a crossing RWRE with random potential, see (2.1). Given this interpretation, one can prove Theorem 1.3 by a similar way to those in [9, 10].

The proof of lower bound is a bit more complicated. The key to the proof is that where $\alpha_\lambda(y)$ is differentiable in λ , we have

$$\tilde{P}_{\omega,\sigma}^0\left(H^Z(ty)/t \in \frac{\partial}{\partial \lambda} \alpha_\lambda(y)(1 - \epsilon, 1)\right) \geq \exp\{-t(\alpha_\lambda(y) - \lambda) + o(t)\}$$

for any $\epsilon > 0$, where $H^Z(ty)$ denotes the first time for Z to hit $[ty]$. This means that we know the cost for the random walk to make a crossing in the speed $1/\frac{\partial}{\partial \lambda} \alpha_\lambda(y)$. In particular, if $\frac{\partial}{\partial \lambda} \alpha_\lambda(y) = 1$ for some λ , then the above is already quite close to the lower bound since such a λ maximizes $\alpha_\lambda(y) - \lambda$. However we do not know if this is the case in general since

- (1) it is hard to check the differentiability of $\alpha_\lambda(y)$ and
- (2) it may happen that $\alpha'_{\lambda+}(y)|_{\lambda=0} = \sup_{\lambda \geq 0} \alpha'_{\lambda+}(y) < 1$.

To circumvent the first issue, we choose two differentiability points close above and below a maximizer of $\lambda \mapsto \alpha_\lambda(y) - \lambda$. Then, we let the walker move slower than needed toward an intermediate point and faster on the rest of the way to $[ty]$ to achieve the expected speed, see Lemma 4.2. As for the second issue, we find a *trap* around $[ty]$, that is, a region where the walker can spend time with relatively high probability. This is the content of Lemma 4.3 and it is here where we need the nestling assumption.

1.2. Notation. For $x = (x_1, \dots, x_d)$ in \mathbb{R}^d , we write $|x| := |x_1| + \dots + |x_d|$. Open l_1 -balls with center $x \in \mathbb{R}^d$ and radius $r \geq 0$ are denoted by $B(x, r)$ and closed balls by $\overline{B}(x, r)$. We write $[x]$ for a lattice site with minimal l_1 -distance from x chosen by some deterministic rule. Note that always $|x - [x]| \leq d/2$. Similarly, let $[A] := \{[x]; x \in A\}$ for each subset $A \subset \mathbb{R}^d$.

2. LYAPUNOV EXPONENTS

In this section, we show Theorem 1.2. We start with the triangle inequality and integrability properties for a_λ . To this end, let $H^X(y) := \inf\{n \geq 0; X_n = y\}$ be the first passage time through y for the random walk $(X_n)_{n=0}^\infty$. Then

$$H^Z(y) = \sum_{m=0}^{H^X(y)-1} \tau_m(X_m)$$

holds on the event $\{H^X(y) < \infty\} = \{H^Z(y) < \infty\}$ and hence by Fubini's theorem,

$$(2.1) \quad e_\lambda(x, y, \omega, \sigma) = E_\omega^x \left[\exp \left\{ - \sum_{m=0}^{H^X(y)-1} \theta_{\lambda, \sigma}(X_m) \right\} \mathbb{1}_{\{H^X(y) < \infty\}} \right].$$

Lemma 2.1 For any $\lambda \geq 0$, $x, y, z \in \mathbb{Z}^d$, $\omega \in \Omega$ and $\sigma \in \Sigma$,

$$(2.2) \quad a_\lambda(x, y, \omega, \sigma) \leq a_\lambda(x, z, \omega, \sigma) + a_\lambda(z, y, \omega, \sigma).$$

Moreover, if $d = 1$ and $x \leq z \leq y$ or $y \leq z \leq x$, then equality holds in (2.2).

Proof. Set $H_z^X(y) := \inf\{n \geq H^X(z); X_n = y\}$. Using the strong Markov property, we have

$$(2.3) \quad \begin{aligned} e_\lambda(x, y, \omega, \sigma) &\geq E_\omega^x \left[\exp \left\{ - \sum_{n=0}^{H_z^X(y)-1} \theta_{\lambda, \sigma}(X_n) \right\} \mathbb{1}_{\{H_z^X(y) < \infty\}} \right] \\ &= e_\lambda(x, z, \omega, \sigma) e_\lambda(z, y, \omega, \sigma). \end{aligned}$$

By taking logarithm, this proves (2.2). If $d = 1$ and $x \leq z \leq y$ or $y \leq z \leq x$, then equality holds in (2.3) since the random walk $(X_n)_{n=0}^\infty$ has to go through z before reaching y . \square

Lemma 2.2 Let $\lambda \geq 0$ and $p \geq 1$. Then $a_\lambda(0, x, \omega, \sigma) \in L^d(\mathbb{P} \otimes \mathbf{P})$ holds for $x \in \mathbb{Z}^d$. Moreover, the collection of random variables $a_\lambda(0, x, \sigma, \omega)/|x|$, $x \in \mathbb{Z}^d \setminus \{0\}$ is uniformly integrable under $\mathbb{P} \otimes \mathbf{P}$ and we have for all $x \in \mathbb{Z}^d$,

$$(2.4) \quad c_1(\lambda)|x| \leq \mathbb{E} \otimes \mathbf{E}[a_\lambda(0, x, \omega, \sigma)] \leq c_2(\lambda)|x|,$$

where

$$\begin{aligned} c_1(\lambda) &:= -\log \mathbf{E}[\exp\{-\theta_{\lambda, \sigma}(0)\}], \\ c_2(\lambda) &:= \max_{|e|=1} \mathbb{E}[-\log \omega(0, e)] + \mathbf{E}[\theta_{\lambda, \sigma}(0)]. \end{aligned}$$

Proof. Let $x \in \mathbb{Z}^d \setminus \{0\}$. By forcing the walker to follow a nearest neighbor path $(0 = r_0, r_1, \dots, r_m = x)$ from 0 to x with minimal length $m = |x|$, we have

$$e_\lambda(0, x, \omega, \sigma) \geq \exp \left\{ - \sum_{n=0}^{m-1} \theta_{\lambda, \sigma}(r_n) \right\} \prod_{n=0}^{m-1} \omega(r_n, r_{n+1} - r_n).$$

It follows that

$$(2.5) \quad \frac{a_\lambda(0, x, \omega, \sigma)}{|x|} \leq -\frac{1}{m} \sum_{n=0}^{m-1} \log \omega(r_n, r_{n+1} - r_n) + \frac{1}{m} \sum_{n=0}^{m-1} \theta_{\lambda, \sigma}(r_n)$$

and hence $a_\lambda(0, x, \omega, \sigma) \in L^d(\mathbb{P} \otimes \mathbf{P})$ by Assumption 1.1-(1). Moreover, Jensen's inequality implies that for any $\gamma \geq 0$,

$$(2.6) \quad \begin{aligned} & \mathbb{E} \otimes \mathbf{E} \left[\left(-\frac{1}{m} \sum_{n=0}^{m-1} \log \omega(r_n, r_{n+1} - r_n) + \frac{1}{m} \sum_{n=0}^{m-1} \theta_{\lambda, \sigma}(r_n) - \gamma \right)_+ \right] \\ & \leq \frac{1}{m} \sum_{n=0}^{m-1} \mathbb{E} \otimes \mathbf{E} [(-\log \omega(r_n, r_{n+1} - r_n) + \theta_{\lambda, \sigma}(r_n) - \gamma)_+] \\ & \leq \max_{|e|=1} \mathbb{E} \otimes \mathbf{E} [(-\log \omega(0, e) + \theta_{\lambda, \sigma}(0) - \gamma)_+]. \end{aligned}$$

By (2.5) and (2.6), we have for any $\gamma \geq 0$,

$$\begin{aligned} & \limsup_{M \rightarrow \infty} \sup_{x \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E} \otimes \mathbf{E} \left[\frac{a_\lambda(0, x, \omega, \sigma)}{|x|} \mathbb{1}_{\{a_\lambda(0, x, \omega, \sigma)/|x| > M\}} \right] \\ & \leq \limsup_{M \rightarrow \infty} \sup_{x \in \mathbb{Z}^d \setminus \{0\}} \left\{ \gamma \mathbb{P} \otimes \mathbf{P} \left(\frac{a_\lambda(0, x, \omega, \sigma)}{|x|} > M \right) \right. \\ & \quad \left. + \mathbb{E} \otimes \mathbf{E} \left[\left(\frac{a_\lambda(0, x, \omega, \sigma)}{|x|} - \gamma \right)_+ \right] \right\} \\ & \leq \limsup_{M \rightarrow \infty} \frac{\gamma}{M} c_2(\lambda) + \max_{|e|=1} \mathbb{E} \otimes \mathbf{E} [(-\log \omega(0, e) + \theta_{\lambda, \sigma}(0) - \gamma)_+] \\ & = \max_{|e|=1} \mathbb{E} \otimes \mathbf{E} [(-\log \omega(0, e) + \theta_{\lambda, \sigma}(0) - \gamma)_+]. \end{aligned}$$

Since $\log \min_{|e|=1} \omega(0, e) \in L^1(\mathbb{P})$ and $\sigma(0) \in L^1(\mathbf{P})$, $\mathbb{E} \otimes \mathbf{E} [(-\log \omega(0, e) + \theta_{\lambda, \sigma}(0) - \gamma)_+]$ tends to zero as $\gamma \rightarrow \infty$ for any $e \in \mathbb{Z}^d$ with $|e| = 1$ by Lebesgue's dominated convergence theorem. We thereby find that the collection of random variables $a_\lambda(0, x, \omega, \sigma)/|x|$, $x \in \mathbb{Z}^d \setminus \{0\}$ is uniformly integrable under $\mathbb{P} \otimes \mathbf{P}$.

Finally, we show (2.4). The second inequality of (2.4) follows from (2.5) and (2.6) by taking $\gamma = 0$. To show the other inequality, we introduce for $\lambda \geq 0$, $\omega \in \Omega$, $\sigma \in \Sigma$ and $x, y \in \mathbb{Z}^d$ the path measure

$$\widehat{P}_{\lambda, \omega, \sigma}^{x, y}(dX.) := e_\lambda(x, y, \omega, \sigma)^{-1} \exp \left\{ \sum_{n=0}^{H^X(y)-1} \theta_{\lambda, \sigma}(X_n) \right\} \mathbb{1}_{\{H^X(y) < \infty\}} P_\omega^x(dX.)$$

and denote its expectation by $\widehat{E}_{\lambda,\omega,\sigma}^{x,y}$. In addition, let us define $\#\mathcal{A}(x, X.) := \{X_0, \dots, X_{H^X(x)}\}$. Since $\#\mathcal{A}(x, X.) \geq |x|$ P_ω^0 -a.s., we get from Jensen's inequality that

$$\begin{aligned} c_1(\lambda)|x| &\leq \mathbb{E} \otimes \mathbf{E} \left[\log \widehat{E}_{\lambda,\omega,\sigma}^{0,x} [\exp\{c_1(\lambda) \#\mathcal{A}(x, X.)\}] \right] \\ &\leq \mathbb{E} \otimes \mathbf{E} [a_\lambda(0, x, \omega, \sigma)] + \log \mathbb{E} \otimes E_\omega^0 \left[\prod_{y \in \mathcal{A}(x, X.)} \mathbf{E} [\exp\{c_1(\lambda) - \theta_{\lambda,\sigma}(y)\}] \right] \\ &= \mathbb{E} \otimes \mathbf{E} [a_\lambda(0, x, \omega, \sigma)], \end{aligned}$$

where the last equality is due to the choice of $c_1(\lambda)$. This proves the first and second inequalities of (2.4). \square

Now we are in position to prove Theorem 1.2.

Proof of Theorem 1.2. Given Lemma 2.1 and 2.2, the proof goes in the same line as that of [9, Proposition 4] or [10, Proposition 3]. Namely, we first prove (1.1) by using the subadditive ergodic theorem. Then $\alpha_\lambda(qx) = q\alpha_\lambda(x)$ follows for $q \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ by stationarity. Finally, we extend $\alpha_\lambda(\cdot)$ to \mathbb{Q}^d by $\alpha_\lambda(x) = \alpha_\lambda(x)/q$ and then to \mathbb{R}^d by continuity. The properties of α_λ as a function of λ follows by that of $\theta_{\lambda,\sigma}(x)$. See the above references for details. \square

3. SHAPE THEOREM

Our goal in this section is to prove Theorem 1.3, which is called the shape theorem, and to derive its generalizations. To this end, we recall the next lemma which plays the role of the maximal lemmas of random walk in a nonnegative random potential and random walk in random environment. The proof is the same as that of [9, Lemma 7] or [10, Lemma 6] and we omit it.

Lemma 3.1 For each $\lambda \geq 0$, there is a positive constant $c_3(\lambda)$ such that the following holds $\mathbb{P} \otimes \mathbf{P}$ -a.s.: for any $\epsilon \in \mathbb{Q} \cap (0, 1)$ there exists a positive number $R = R(\lambda, \epsilon, \omega, \sigma)$ such that

$$(3.1) \quad \sup\{d_\lambda([x], [y], \omega, \sigma); y \in \mathbb{R}^d, |x - y| \leq \epsilon|x|\} < c_3(\lambda)\epsilon|x|$$

holds for all $x \in \mathbb{R}^d$ with $|x| > R$.

Proof of Theorem 1.3. Given Lemma 3.1, one can prove $\mathbb{P} \otimes \mathbf{P}$ -a.s. convergence by the same strategy as in [9, Theorem A]. Then $L^1(\mathbb{P} \otimes \mathbf{P})$ -convergence follows from $\mathbb{P} \otimes \mathbf{P}$ -a.s. convergence by uniform integrability provided by Lemma 2.2. \square

We next consider a generalization of Theorem 1.3 for point-to-set distances instead of point-to-point distances. Let us define $e_\lambda(x, K, \omega, \sigma)$ for a nonempty subset $K \subset \mathbb{R}^d$ as in (2.1) but $H^X(y)$ replaced by $H^X(K) := \inf\{H^X(y); y \in K\}$. Furthermore, we write $a_\lambda(x, K, \omega, \sigma)$ for $-\log e_\lambda(x, K, \omega, \sigma)$ and denote the distance between x and K by $\text{dist}(x, K) := \inf\{|x - y|; y \in K\}$. Given Theorem 1.3, one can prove the following corollary by the same way as [9, Corollary 16].

Corollary 3.2 Let $\lambda \geq 0$ and $(K_n)_{n=1}^\infty$ be a sequence of subsets of \mathbb{R}^d such that $K_n \neq \emptyset$ and $\text{dist}(0, K_n) \rightarrow \infty$. Then,

$$\lim_{n \rightarrow \infty} \frac{a_\lambda(0, [K_n], \omega, \sigma) - \inf_{x \in K_n} \alpha_\lambda(x)}{\text{dist}(0, K_n)} = 0$$

$\mathbb{P} \otimes \mathbf{P}$ -a.s.

Let us finally extend Theorem 1.3 to a directionally uniform version. The shape theorem will be used to relate crossing costs to the Lyapunov exponent in the proof of the large deviation lower bound. However, Theorem 1.3 does not suffice as it is. As we explained in the introduction, we shall divide the crossing into two pieces and for the second piece, we need a shape theorem with *moving starting points*.

Corollary 3.3 Let $x \in \mathbb{Q}^d \setminus \{0\}$ and suppose that $\rho_1, \rho_2 \in \mathbb{R}$ satisfy $0 \leq \rho_1 < \rho_2$. Under the assumption of Theorem 1.3, the following holds $\mathbb{P} \otimes \mathbf{P}$ -a.s.: for all $\lambda \geq 0$ and all sequences $(x_n)_{n=1}^\infty$ of \mathbb{R}^d with $x_n/n \rightarrow x$,

$$\lim_{n \rightarrow \infty} \frac{a_\lambda([\rho_1 x_n], [\rho_2 x_n], \omega, \sigma) - (\rho_2 - \rho_1) \alpha_\lambda(x_n)}{|x_n|} = 0.$$

Proof. By the continuity of $\alpha_\lambda(\cdot)$, it suffices to prove that $\mathbb{P} \otimes \mathbf{P}$ -almost surely,

$$(3.2) \quad \lim_{n \rightarrow \infty} \frac{a_\lambda([\rho_1 x_n], [\rho_2 x_n], \omega, \sigma)}{n} = (\rho_2 - \rho_1) \alpha_\lambda(x)$$

holds for all $\lambda \geq 0$ and all sequences $(x_n)_{n=1}^\infty$ of \mathbb{R}^d with $x_n/n \rightarrow x$. Thank to Lemma 2.1 and Theorem 1.3, we know that the lower bound

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{n} a_\lambda([\rho_1 x_n], [\rho_2 x_n], \omega, \sigma) \\ & \geq \liminf_{n \rightarrow \infty} \frac{1}{n} (a_\lambda(0, [\rho_2 x_n], \omega, \sigma) - a_\lambda(0, [\rho_1 x_n], \omega, \sigma)) \\ & = (\rho_2 - \rho_1) \alpha_\lambda(x) \end{aligned}$$

is valid. To show the upper bound

$$(3.3) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} a_\lambda([\rho_1 x_n], [\rho_2 x_n], \omega, \sigma) \leq (\rho_2 - \rho_1) \alpha_\lambda(x),$$

note that we have for $K \in \mathbb{N}$ with $Kx \in \mathbb{Z}^d$,

$$\begin{aligned} (3.4) \quad a_\lambda([\rho_1 n x], [\rho_2 n x], \omega, \sigma) & \leq \sum_{m=\lceil \rho_1 n/K \rceil}^{\lfloor \rho_2 n/K \rfloor - 1} a_\lambda(m K x, (m+1) K x, \omega, \sigma) \\ & \quad + a_\lambda([\rho_1 n x], \lceil \rho_1 n/K \rceil K x, \omega, \sigma) \\ & \quad + a_\lambda(\lfloor \rho_2 n/K \rfloor K x, [\rho_2 n x], \omega, \sigma). \end{aligned}$$

by Lemma 2.1. Birkhoff's ergodic theorem shows that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=\lceil \rho_1 n/K \rceil}^{\lfloor \rho_2 n/K \rfloor - 1} a_\lambda(mKx, (m+1)Kx, \omega, \sigma) = \frac{\rho_2 - \rho_1}{K} \mathbb{E} \otimes \mathbf{E}[a_\lambda(0, Kx, \omega, \sigma)]$$

holds $\mathbb{P} \otimes \mathbf{P}$ -a.s. On the other hand, we know that for any $\epsilon \in \mathbb{Q} \cap (0, 1)$ and sufficiently large n ,

$$\begin{aligned} \left| [\rho_i nx] - \left\lceil \frac{\rho_i n}{K} \right\rceil Kx \right| &\leq |[\rho_i nx] - \rho_i nx| + \left| \frac{\rho_i n}{K} - \left\lceil \frac{\rho_i n}{K} \right\rceil \right| K|x| \\ &\leq \frac{d}{2} + K|x| \leq \epsilon |[\rho_i nx]| \end{aligned}$$

for $i = 1, 2$. Thus we can apply Lemma 3.1 to show that $\mathbb{P} \otimes \mathbf{P}$ -a.s., the sum of the second and third terms of the right-hand side of (3.4) is smaller than

$$\begin{aligned} &d_\lambda([\rho_1 nx], \lceil \rho_1 n/K \rceil Kx, \omega, \sigma) + d_\lambda([\rho_2 nx], \lfloor \rho_2 n/K \rfloor Kx, \omega, \sigma) \\ &< c_3(\lambda) \epsilon (|[\rho_1 nx]| + |[\rho_2 nx]|). \end{aligned}$$

It follows that

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \frac{1}{n} a_\lambda([\rho_1 nx], [\rho_2 nx], \omega, \sigma) \\ &\leq \frac{\rho_2 - \rho_1}{K} \mathbb{E} \otimes \mathbf{E}[a_\lambda(0, Kx, \omega, \sigma)] + c_3(\lambda) \epsilon (\rho_1 + \rho_2) |x| \end{aligned}$$

and therefore letting $\epsilon \searrow 0$ and $K \rightarrow \infty$, we obtain from Theorem 1.2 that

$$(3.5) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} a_\lambda([\rho_1 nx], [\rho_2 nx], \omega, \sigma) \leq (\rho_2 - \rho_1) \alpha_\lambda(x)$$

holds $\mathbb{P} \otimes \mathbf{P}$ -a.s. Using Lemma 2.1, we have

$$\begin{aligned} &a_\lambda([\rho_1 nx], [\rho_2 nx], \omega, \sigma) - a_\lambda([\rho_1 x_n], [\rho_2 x_n], \omega, \sigma) \\ &\leq a_\lambda([\rho_1 nx], [\rho_1 x_n], \omega, \sigma) + a_\lambda([\rho_2 x_n], [\rho_2 nx], \omega, \sigma) \\ &\leq d_\lambda([\rho_1 nx], [\rho_1 x_n], \omega, \sigma) + d_\lambda([\rho_2 x_n], [\rho_2 nx], \omega, \sigma), \end{aligned}$$

and similarly

$$\begin{aligned} &a_\lambda([\rho_1 nx], [\rho_2 nx], \omega, \sigma) - a_\lambda([\rho_1 x_n], [\rho_2 x_n], \omega, \sigma) \\ &\geq -d_\lambda([\rho_1 nx], [\rho_1 x_n], \omega, \sigma) - d_\lambda([\rho_2 nx], [\rho_2 x_n], \omega, \sigma). \end{aligned}$$

Furthermore, we have for $\epsilon \in \mathbb{Q} \cap (0, 1/2)$ and sufficiently large n ,

$$|[\rho_i nx] - [\rho_i x_n]| \leq 2\epsilon |[\rho_i nx]|, \quad i = 1, 2.$$

Lemma 3.1 thereby implies that

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \frac{1}{n} |a_\lambda([\rho_1 nx], [\rho_2 nx], \omega, \sigma) - a_\lambda([\rho_1 x_n], [\rho_2 x_n], \omega, \sigma)| \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} (d_\lambda([\rho_1 nx], [\rho_1 x_n], \omega, \sigma) + d_\lambda([\rho_2 nx], [\rho_2 x_n], \omega, \sigma)) \end{aligned}$$

$$\leq 2c_3(\lambda)\epsilon(\rho_1 + \rho_2)|x|.$$

holds $\mathbb{P} \otimes \mathbf{P}$ -a.s. This together with (3.5) proves (3.3) since ϵ is arbitrary. \square

Remark 3.4 Zerner proved a stronger version of shape theorem and used it to prove the large deviation lower bound in [10]. We find difficulty in proving a shape theorem in such a general form. Note that our Lyapunov exponent can be regarded as a mixture of those in [9] and [10] and in the former paper, the *uniform shape theorem* requires some assumptions. Our strategy, using the above directionally uniform shape theorem, dates back to Sznitman's work on large deviations for Brownian motion in Poissonian obstacles [7].

4. LARGE DEVIATION ESTIMATES

Our goal in this section is to show Theorem 1.4. We prove upper and lower bounds of Theorem 1.4 in Subsections 4.1 and 4.2 respectively.

4.1. Upper bound. In this subsection, we prove the upper bound (1.3) of Theorem 1.4. Let us first mention some properties of the rate function I . We denote the essential domain of the rate function I by \mathcal{D}_I , that is, $\mathcal{D}_I := \{x \in \mathbb{R}^d; I(x) < \infty\}$. It is easy to see that I is convex on \mathbb{R}^d , lower semicontinuous on \mathcal{D}_I and continuous on the interior of \mathcal{D}_I . Furthermore, by Theorem 1.2,

$$(4.1) \quad 0 \leq I(x) \leq |x| \max_{|e|=1} \mathbb{E}[-\log \omega(0, e)]$$

holds for all $x \in \mathbb{R}^d$ with $|x| \leq \mathbf{E}[\int_0^\infty s \sigma_0(ds)]^{-1}$.

Proof of the upper bound (1.3) in Theorem 1.4. Let us first show that

$$(4.2) \quad \lim_{R \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \tilde{P}_{\omega, \sigma}^0(Z_t \notin t\overline{B}(0, R)) = -\infty.$$

holds $\mathbb{P} \otimes \mathbf{P}$ -a.s. We have for any $\lambda, t \geq 0$ and subset $K \subset \mathbb{R}^d$,

$$\begin{aligned} \tilde{P}_{\omega, \sigma}^0(Z_t \notin tK) &\leq \exp\{\lambda t\} \tilde{E}_{\omega, \sigma}^0[\exp\{-\lambda t\} \mathbb{1}_{\{H^Z([tK]) \leq t\}}] \\ &\leq \exp\{\lambda t\} e_\lambda(0, [tK], \omega, \sigma), \end{aligned}$$

and hence Corollary 3.2 implies that

$$(4.3) \quad \begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \tilde{P}_{\omega, \sigma}^0(Z_t \in tK) &\leq \limsup_{t \rightarrow \infty} \left(\lambda - \frac{1}{t} \log a_\lambda(0, [tK], \omega, \sigma) \right) \\ &= \lambda - \inf_{x \in K} \alpha(x) \end{aligned}$$

holds $\mathbb{P} \otimes \mathbf{P}$ -a.s. It follows from Theorem 1.2 that we obtain $\mathbb{P} \otimes \mathbf{P}$ -a.s.,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \tilde{P}_{\omega, \sigma}^0(Z_t \notin t\overline{B}(0, R)) &\leq - \inf_{x \in \overline{B}(0, R)^c} (\alpha_\lambda(x) - \lambda) \\ &\leq - \inf_{x \in \overline{B}(0, R)^c} (|x|(-\log \mathbf{E}[\exp\{-\theta_{\lambda, \sigma}(0)\}]) - \lambda) \\ &\leq -R(-\log \mathbf{E}[\exp\{-\theta_{\lambda, \sigma}(0)\}]) + \lambda, \end{aligned}$$

which proves (4.2) by letting $R \rightarrow \infty$.

Now we show the upper bound (1.3). It suffices to consider compact $A \subset \mathbb{R}^d$ thanks to (4.2). Moreover, we may assume $0 \notin A$ since $\inf_{x \in A} I(x) = 0$ if $0 \in A$ by (4.1). For every $\delta > 0$ we introduce the δ -rate function I^δ as

$$I^\delta(x) := (I(x) - \delta) \wedge \frac{1}{\delta}$$

and set

$$A_\lambda(\delta) := \left\{ y \in A; \alpha_\lambda(y) - \lambda > \inf_{x \in A} I^\delta(x) - \delta \right\}$$

for each $\lambda \geq 0$. Applying (4.3) for $K = A_\lambda(\delta)$, we obtain from Corollary 3.2 that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \tilde{P}_{\omega, \sigma}^0(Z_t \in tA_\lambda(\delta)) \leq \lambda - \inf_{x \in A_\lambda(\delta)} \alpha(x) \leq \delta - \inf_{x \in A} I^\delta(x).$$

Since $A = \bigcup_{\lambda \geq 0} A_\lambda(\delta)$ and A is compact, there are λ_i ($1 \leq i \leq m$) such that $A_{\lambda_i}(\delta)$ ($1 \leq i \leq m$) cover A . Thus, for any $\delta > 0$,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \tilde{P}_{\omega, \sigma}^0(Z_t \in tA) &\leq \max_{1 \leq i \leq m} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \tilde{P}_{\omega, \sigma}^0(Z_t \in tA_{\lambda_i}(\delta)) \\ &\leq \delta - \inf_{x \in A} I^\delta(x). \end{aligned}$$

Since $\lim_{\delta \searrow 0} \inf_{x \in A} I^\delta(x) = \inf_{x \in A} I(x)$, the upper bound (1.3) follows by letting $\delta \searrow 0$. \square

4.2. Lower bound. In this subsection, we prove the lower bound (1.4) of Theorem 1.4. Let us start with the following lemma.

Lemma 4.1 Let $x \in \mathbb{Q}^d \setminus \{0\}$ and assume that $\rho_1, \rho_2 \in \mathbb{R}$ satisfy $0 \leq \rho_1 < \rho_2$. Then the following holds $\mathbb{P} \otimes \mathbf{P}$ -a.s.:

$$(4.4) \quad \lim_{t \rightarrow \infty} \hat{P}_{\lambda, \omega, \sigma}^{[\rho_1 x_t], [\rho_2 x_t]} \left(\frac{H^Z([\rho_2 x_t])}{(\rho_2 - \rho_1)t} \in (\gamma_1, \gamma_2) \right) = 1$$

for all sequences $(x_t)_{t \geq 0}$ of \mathbb{R}^d with $x_t/t \rightarrow x$ as $t \rightarrow \infty$ and all $\lambda > 0$, $\gamma_1, \gamma_2 \in \mathbb{R}$ satisfying

$$(4.5) \quad 0 \leq \gamma_1 < \alpha'_{\lambda+}(x) \leq \alpha'_{\lambda-}(x) < \gamma_2.$$

Proof. Corollary 3.3 implies that $\mathbb{P} \otimes \mathbf{P}$ -a.s., for $x, \rho_1, \rho_2, (x_t)_{t \geq 0}, \lambda, \gamma_1, \gamma_2$ as above and $\mu \in (0, \lambda)$,

$$\begin{aligned} &\limsup_{t \rightarrow \infty} \frac{1}{t} \log \hat{P}_{\lambda, \omega, \sigma}^{[\rho_1 x_t], [\rho_2 x_t]}(H^Z([x_t]) \geq (\rho_2 - \rho_1)\gamma_2 t) \\ &= (\rho_2 - \rho_1)\alpha_\lambda(x) \\ &\quad + \limsup_{t \rightarrow \infty} \frac{1}{t} \log \tilde{E}_{\omega, \sigma}^{[\rho_1 x_t]} [\exp\{(-\lambda + \mu)H^Z([\rho_2 x_t])\} \exp\{-\mu H^Z([\rho_2 x_t])\}] \\ &\quad \times \mathbb{1}_{\{H^Z([\rho_2 x_t]) < \infty, H^Z([\rho_2 x_t]) \geq (\rho_2 - \rho_1)\gamma_2 t\}}] \end{aligned}$$

$$\begin{aligned}
&\leq (\rho_2 - \rho_1)\alpha_\lambda(x) - \mu(\rho_2 - \rho_1)\gamma_2 - (\rho_2 - \rho_1)\alpha_{\lambda-\mu}(x) \\
&= \mu(\rho_2 - \rho_1) \left(\frac{\alpha_\lambda(x) - \alpha_{\lambda-\mu}(x)}{\mu} - \gamma_2 \right).
\end{aligned}$$

It follows from (4.5) that the most right-hand side of the above expression is negative for μ small enough. Since the corresponding statement can be proved for the event $\{H^Z([x_t]) \leq (\rho_2 - \rho_1)\gamma_1 t\}$ in the same manner, we have (4.4). \square

Proof of the lower bound (1.4) in Theorem 1.4. It suffices to show that $\mathbb{P} \otimes \mathbf{P}$ -a.s.,

$$(4.6) \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \log \tilde{P}_{\omega, \sigma}^0(Z_t \in tB(z, r)) \geq -I(z)$$

holds for all $z \in \mathbb{Q}^d \setminus \{0\} \cap \mathcal{D}_I$ and $0 < r \in \mathbb{Q}$. To prove this, let us define

$$\lambda_\infty := \sup\{\lambda > 0; \alpha'_{\lambda-}(z) \geq 1\},$$

with the convention $\sup \emptyset = 0$. It is easy to check that $I(z) = \alpha_{\lambda_\infty}(z) - \lambda_\infty$ in the case $\lambda_\infty < \infty$, and $I(z) = \lim_{\lambda \rightarrow \infty} (\alpha_\lambda(z) - \lambda)$ otherwise. We first treat the case $\lambda_\infty < \infty$. By the concavity of $\alpha_\lambda(z)$ in λ , we can find sequences $(\gamma_n)_{n=1}^\infty$, $(\delta_n)_{n=1}^\infty$ and $(\lambda_n)_{n=1}^\infty$ such that

- if $\alpha'_{\lambda-}(z) < 1$ for all $\lambda > 0$, then $\alpha'_{\lambda_n}(z)$ exists, $\lambda_n \rightarrow 0$ and

$$\gamma_n := \alpha'_{\lambda_n}(z) \left(1 - \frac{1}{n}\right), \quad \delta_n := \alpha'_{\lambda_n}(z) \left(1 + \frac{1 - \alpha'_{\lambda_n}(z)}{n}\right) < 1,$$

- otherwise, $\lambda_n \rightarrow \lambda_\infty$, $1 - 2/n \in [\alpha'_{\lambda_n+}(z), \alpha'_{\lambda_n-}(z)]$ and

$$\gamma_n := 1 - \frac{3}{n}, \quad \delta_n := 1 - \frac{1}{n}.$$

Observe that for the above sequences, we have

$$(4.7) \quad (\gamma_n, \delta_n) \cap [\alpha'_{\lambda_n+}(z), \alpha'_{\lambda_n-}(z)] \neq \emptyset.$$

Now recall that Assumption 1.1-(2) is equivalent to the following (see [10, Proposition 8]): for all $\epsilon > 0$ there is some $R(\epsilon) \geq 2$ such that

$$(4.8) \quad \mathbb{P}(P_\omega^0(X_{R(\epsilon)} = 0) > e^{-\epsilon R(\epsilon)}) > 0.$$

We choose $R(\epsilon) > 0$ satisfying (4.8) for $\epsilon > 0$ and fix $\delta > 0$ with $\mathbf{P}(\theta_{1,\sigma}(0) \geq \delta) > 0$. Then, for $y \in (2R(\epsilon) + 1)\mathbb{Z}^d$

$$\Phi_0(y, \epsilon) := \left\{ (\omega, \sigma) : P_\omega^y(X_{R(\epsilon)} = y) > e^{-\epsilon R(\epsilon)}, \min_{x \in B(y, R(\epsilon))} \theta_{1,\sigma}(x) \geq \delta \right\}.$$

has strictly positive $\mathbb{P} \otimes \mathbf{P}$ -probability and independent for different y 's. Let $y_t = y_t(\epsilon, \omega, \sigma) \in (2R(\epsilon) + 1)\mathbb{Z}^d$ be a vertex with minimal distance from $[tz]$ such that $(\omega, \sigma) \in \Phi_0(y_t, \epsilon)$. A simple application of the Borel–Cantelli lemma shows that $\mathbb{P} \otimes \mathbf{P}$ -a.s., these y_t exist and satisfy

$$(4.9) \quad |tz - y_t| \leq 2(\log t)^2$$

for all sufficiently large t . Let us introduce some more notations. Denote

$$T_n^X(\epsilon, t) := \inf \left\{ k \geq 0; \sum_{m=0}^{k-1} \tau_m(X_m) > \gamma_n t, X_k = y_t \right\}$$

and

$$T_n^Z(\epsilon, t) := \inf \{ s > \gamma_n t; Z_s = y_t \} = \sum_{m=0}^{T_n^X(t)-1} \tau_m(X_m).$$

We then define the random variable

$$b_n(\epsilon, t, \omega, \sigma) := -\log \tilde{E}_{\omega, \sigma}^0 [\exp \{ -\lambda_n T_n^Z(\epsilon, t) \} \mathbb{1}_{\{T_n^Z(\epsilon, t) < \delta_n t\}}]$$

and the event

$$\Lambda_1(n, \epsilon, t) := \{ (X, \tau) : Z_{s+T_n^Z(\epsilon, t)} \in B(y_t, R(\epsilon)) \text{ for all } s \in [0, (1 - \gamma_n)t] \}.$$

Now, the left-hand side of (4.6) is greater than

$$\lambda_n \gamma_n + \liminf_{t \rightarrow \infty} \frac{1}{t} \log \tilde{E}_{\omega, \sigma}^0 [\exp \{ -\lambda_n T_n^Z(\epsilon, t) \} \mathbb{1}_{\{T_n^Z(\epsilon, t) < \delta_n t\} \cap \Lambda_1(n, \epsilon, t)}]$$

since $B(y_t, R(\epsilon)) \subset tB(z, r)$ for sufficiently large t by (4.9). The strong Markov property shows that the above expression equals to

$$(4.10) \quad \lambda_n \gamma_n - \limsup_{t \rightarrow \infty} \frac{1}{t} b_n(t, \epsilon, \omega, \sigma) + \liminf_{t \rightarrow \infty} \frac{1}{t} \log \sum_{\ell=0}^{\infty} \tilde{P}_{\omega, \sigma}^{y_t}(\Lambda_2(n, \epsilon, t, \ell)),$$

where $\Lambda_2(n, \epsilon, t, \ell)$ is the event defined as

$$\Lambda_2(n, \epsilon, t, \ell) := \left\{ (X, \tau) : \sum_{m=0}^{\ell-1} \tau_m(X_m) \leq (1 - \gamma_n)t < \sum_{m=0}^{\ell} \tau_m(X_m), \right. \\ \left. X_m \in B(y_t, R(\epsilon)) \text{ for all } m \in [0, \ell - 1] \right\}.$$

To control the second and third term of (4.10), we use the following two lemmas.

Lemma 4.2 For any $\epsilon > 0$ and $n \geq 1$, we have $\mathbb{P} \otimes \mathbf{P}$ -a.s.,

$$\lim_{t \rightarrow \infty} \frac{1}{t} b_n(\epsilon, t, \omega, \sigma) = \alpha_{\lambda_n}(z).$$

Lemma 4.3 For any $\epsilon > 0$ and $n \geq 1$, we have $\mathbb{P} \otimes \mathbf{P}$ -a.s.,

$$(4.11) \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \log \sum_{\ell=0}^{\infty} \tilde{P}_{\omega, \sigma}^{y_t}(\Lambda_2(n, \epsilon, t, \ell)) \geq -\frac{2\epsilon}{\delta}.$$

Let us postpone the proofs of these lemmas to the end of this subsection. It follows from Lemma 4.2, 4.3 and (4.10) that $\mathbb{P} \otimes \mathbf{P}$ -a.s.,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log \tilde{P}_{\omega, \sigma}^0(Z_t \in tB(z, r)) \geq \lambda_n \gamma_n - \alpha_{\lambda_n}(z) - \frac{2\epsilon}{\delta},$$

which completes the proof of (4.6) in the case $\lambda_\infty < \infty$ by letting $\epsilon \searrow 0$ and $n \rightarrow \infty$.

We next treat the case $\lambda_\infty = \infty$. In this case, $\alpha'_{\lambda_-}(uz) = u\alpha'_{\lambda_-}(z) < 1$ holds for all $u \in \mathbb{Q} \cap (0, 1)$ and all sufficiently large λ . Moreover, for $u \in (0 \vee (1 - r/|z|), 1)$ we pick $0 < r'(u) \in \mathbb{Q}$ with $B(ux, r'(u)) \subset B(x, r)$. Applying the same argument as in the case $\lambda_\infty < \infty$ and using the convexity of the rate function I , one can show that

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \frac{1}{t} \log \tilde{P}_{\omega, \sigma}^0(Z_t \in tB(z, r)) \\ & \geq \liminf_{t \rightarrow \infty} \frac{1}{t} \log \tilde{P}_{\omega, \sigma}^0(Z_t \in tB(\rho z, r'(\rho))) \geq -I(\rho z) \geq -\rho I(z) \end{aligned}$$

holds $\mathbb{P} \otimes \mathbf{P}$ -a.s. This proves (4.6) by letting $\rho \nearrow 1$. \square

We close this section with the proof of Lemma 4.2 and 4.3.

Proof of Lemma 4.2. Note that $b_n(\epsilon, t, \omega, \sigma) \geq a_{\lambda_n}(0, y_t, \omega, \sigma)$ since $H^Z(y_t) \leq T_n^Z(t)$. Theorem 1.3 hence implies that we have $\mathbb{P} \otimes \mathbf{P}$ -a.s.,

$$\alpha_{\lambda_n}(z) \leq \liminf_{t \rightarrow \infty} \frac{1}{t} b_n(\epsilon, t, \omega, \sigma).$$

It remains to show that

$$(4.12) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} b_n(\epsilon, t, \omega, \sigma) \leq \alpha_{\lambda_n}(z)$$

holds $\mathbb{P} \otimes \mathbf{P}$ -a.s. Thanks to (4.7), we can pick $\rho \in (0, 1)$ and $\eta > 0$ such that

$$\rho \alpha'_{\lambda_n+}(z) + (1 - \rho) \alpha'_{\lambda_n-}(z) + [-\eta, \eta] \subset (\gamma_n, \delta_n).$$

Setting $\xi_t := [\rho y_t]$, we know from the choice of ρ and η that $T_n^Z(t) < \delta_n t$ holds on

$$\begin{aligned} \Lambda_3(n, \epsilon, t) := & \left\{ (X, \tau) : \frac{1}{\rho t} \sum_{m=0}^{H^X(\xi_t)-1} \tau_m(X_m) \in \alpha'_{\lambda_n+}(z) + [-\eta, \eta], \right. \\ & \left. \frac{1}{(1 - \rho)t} \sum_{m=H^X(\xi_t)}^{H^X(y_t)-1} \tau_m(X_m) \in \alpha'_{\lambda_n-}(z) + [-\eta, \eta] \right\}. \end{aligned}$$

It follows from this and the strong Markov property that

$$\begin{aligned} & \tilde{E}_{\omega, \sigma}^0[\exp\{-\lambda_n T_n^Z(\epsilon, t)\} \mathbb{1}_{\{T_n^Z(\epsilon, t) < \delta_n t\}}] \\ & \geq \tilde{E}_{\omega, \sigma}^0 \left[\exp \left\{ -\lambda_n \sum_{m=0}^{H^X(y_t)-1} \tau_m(X_m) \right\} \mathbb{1}_{\Lambda_3(n, \epsilon, t)} \right] \end{aligned}$$

$$\begin{aligned}
&= \tilde{E}_{\omega, \sigma}^0 [\exp\{-\lambda_n H^Z(\xi_t)\} \mathbb{1}_{\{H^Z(\xi_t)/(\rho t) \in \alpha'_{\lambda_n+}(z) + [-\eta, \eta]\}}] \\
&\quad \times \tilde{E}_{\omega, \sigma}^{\xi_t} [\exp\{-\lambda_n H^Z(y_t)\} \mathbb{1}_{\{H^Z(y_t)/((1-\rho)t) \in \alpha'_{\lambda_n-}(z) + [-\eta, \eta]\}}].
\end{aligned}$$

Let μ_1 and μ_2 be such that $0 < \mu_1 < \lambda_n < \mu_2$ and $\alpha'_{\lambda_n+}(z) - \eta < \alpha'_{\mu_1+}(z) < \alpha'_{\mu_2-}(z) < \alpha'_{\lambda_n-}(z) + \eta$. Then the most right-hand side of the above expression is bigger than

$$\begin{aligned}
&e_{\mu_2}(0, \xi_t, \omega, \sigma) \hat{P}_{\mu_2, \omega, \sigma}^{0, \xi_t} \left(\frac{H^Z(\xi_t)}{\rho t} \in \alpha'_{\lambda_n+}(z) + [-\eta, \eta] \right) \\
&\quad \times \exp\{-(\lambda_n - \mu_1)(\alpha'_{\lambda_n-}(z) + \eta)(1 - \rho)t\} \\
&\quad \times e_{\mu_1}(\xi_t, y_t, \omega, \sigma) \hat{P}_{\mu_1, \omega, \sigma}^{\xi_t, y_t} \left(\frac{H^Z(y_t)}{(1 - \rho)t} \in \alpha'_{\lambda_n-}(z) + [-\eta, \eta] \right).
\end{aligned}$$

We thereby obtain for $t > 0$,

$$\begin{aligned}
\frac{1}{t} b_n(\epsilon, t, \omega, \sigma) &\leq \frac{1}{t} a_{\lambda_2}(0, \xi_t, \omega, \sigma) - \frac{1}{t} \log \hat{P}_{\mu_2, \omega, \sigma}^{0, \xi_t} \left(\frac{H^Z(\xi_t)}{\rho t} \in \alpha'_{\lambda_n+}(z) + [-\eta, \eta] \right) \\
&\quad + (\lambda_n - \mu_1)(\alpha'_{\lambda_n-}(z) + \eta) + \frac{1}{t} a_{\lambda_1}(\xi_t, y_t, \omega, \sigma) \\
&\quad - \frac{1}{t} \log \hat{P}_{\lambda_1, \omega, \sigma}^{\xi_t, y_t} \left(\frac{H^Z(y_t)}{(1 - \rho)t} \in \alpha'_{\lambda_n-}(z) + [-\eta, \eta] \right).
\end{aligned}$$

Note that we have $y_t/t \rightarrow z$ from (4.9). Therefore, applying Corollary 3.3 and Lemma 4.1, we get $\mathbb{P} \otimes \mathbf{P}$ -a.s.,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} b_n(\epsilon, t, \omega, \sigma) \leq \rho \alpha_{\mu_2}(z) + (\lambda_n - \mu_1)(\alpha'_{\lambda_n-}(z) + \eta) + (1 - \rho) \alpha_{\mu_1}(z),$$

which concludes (4.12) by letting $\mu_1 \nearrow \lambda_n$ and $\mu_2 \searrow \lambda_n$. \square

Proof of Lemma 4.3. Let $L(n, t) := \lfloor 2t/\delta \rfloor + 1$. If $\sum_{m=0}^{L(n, t)-1} \tau_m(X_m) \geq \delta L(n, t)/2$, then $\sum_{m=0}^{\ell-1} \tau_m(X_m) > (1 - \gamma_n)t$ holds for all $\ell \geq L(n, t)$. Thus it follows that

$$\begin{aligned}
&\sum_{\ell=0}^{\infty} \tilde{P}_{\omega, \sigma}^{y_t}(\Lambda_2(n, \epsilon, t, \ell)) \\
&\geq \sum_{\ell=0}^{L(n, t)-1} \tilde{P}_{\omega, \sigma}^{y_t} \left(\sum_{m=0}^{\ell-1} \tau_m(X_m) \leq (1 - \gamma_n)t < \sum_{m=0}^{\ell} \tau_m(X_m), \right. \\
&\quad \left. \sum_{m=0}^{L(n, t)-1} \tau_m(X_m) \geq \frac{\delta}{2} L(n, t), \right. \\
&\quad \left. X_m \in B(\bar{y}_t, R(\epsilon)) \text{ for all } m \in [0, L(n, t) - 1] \right)
\end{aligned}$$

$$\begin{aligned}
&= \tilde{P}_{\omega, \sigma}^{y_t} \left(\sum_{m=0}^{L(n,t)-1} \tau_m(X_m) \geq \frac{\delta}{2} L(n, t), X_k \in B(y_t, R(\epsilon)) \text{ for all } k \in [0, L(n, t) - 1] \right) \\
&= E_{\omega}^{y_t} \left[P_{\sigma}^{\text{HT}} \left(\sum_{m=0}^{L(n,t)-1} \tau_m(X_m) \geq \frac{\delta}{2} L(n, t) \right) \mathbb{1}_{\{X_k \in B(y_t, R(\epsilon)) \text{ for all } k \in [0, L(n, t) - 1]\}} \right].
\end{aligned}$$

On the other hand, the choice of y_t and Chebyshev's inequality show that

$$\begin{aligned}
P_{\sigma}^{\text{HT}} \left(\sum_{m=0}^{L(n,t)-1} \tau_m(X_m) \geq \frac{\delta}{2} L(n, t) \right) &\geq 1 - e^{(\delta/2)L(n,t)} E_{\sigma}^{\text{HT}} \left[\exp \left\{ - \sum_{m=0}^{L(n,t)-1} \tau_m(X_m) \right\} \right] \\
&\geq 1 - e^{(\delta/2)L(n,t)} \prod_{m=0}^{L(n,t)-1} \exp \{ -\theta_{1,\sigma}(X_m) \} \\
&\geq 1 - e^{-(\delta/2)L(n,t)}
\end{aligned}$$

uniformly in paths $(X_n)_{n=0}^{\infty}$ with $X_m \in B(y_t, R(\epsilon))$ for all $m \in [0, L(n, t) - 1]$. It follows from (4.8) that the left-hand side of (4.11) is bigger than

$$\begin{aligned}
&\liminf_{t \rightarrow \infty} \frac{1}{t} \left(\log(1 - e^{-(\delta/2)L(n,t)}) + \log P_{\omega}^{y_t}(X_{R(\epsilon)} = y_t)^{L(n,t)/R(\epsilon)} \right) \\
&\geq \liminf_{t \rightarrow \infty} \left(\frac{1}{t} \log(1 - e^{-(\delta/2)L(n,t)}) - \frac{2\epsilon}{\delta} \right) \\
&= -\frac{2\epsilon}{\delta}.
\end{aligned}$$

Since δ is fixed and ϵ is arbitrary, this proves the lemma. \square

5. FIRST PASSAGE PERCOLATION

In this section, we relate our Lyapunov exponent to the so-called *time constant* of a first passage percolation in the limit $\lambda \rightarrow \infty$. This will be used in the next section to study the asymptotics of the rate function. Throughout this section, we assume that for some deterministic function $L(\lambda)$ with $\lim_{\lambda \rightarrow \infty} L(\lambda) = \infty$,

$$(5.1) \quad \lim_{\lambda \rightarrow \infty} \frac{\theta_{\lambda, \sigma}(z)}{L(\lambda)} = \Theta_{\sigma}(z) \in (0, \infty) \text{ exists } \mathbf{P}\text{-a.s.}$$

As the following examples show, this is a rather restrictive assumption.

Example 5.1 Let us denote the distribution function of $\sigma(0)$ by F_{σ} .

- (1) If $\inf \text{supp} \sigma(0) > 0$ for each σ , then (5.2) holds with $L(\lambda) = \lambda$.
- (2) If for each σ there is $\gamma(\sigma)$ such that $\lim_{x \downarrow 0} F_{\sigma}(x)/x^{\gamma(\sigma)} \in (0, \infty)$, then (5.2) holds with $L(\lambda) = \log \lambda$. This includes the case where all $\{\sigma_x\}_{x \in \mathbb{Z}^d}$ are exponential distribution, which is sometimes called the “random hopping time dynamics”.

- (3) If $x^{\gamma(\sigma)}$ in the previous example is replaced by $x^{\gamma(\sigma)}(\log x)^{\delta(\sigma)}$ with some non-constant $\delta(\sigma)$, then (5.2) fails to hold.
- (4) If there exists $\gamma > 0$ such that $-\lim_{x \downarrow 0} \log F_\sigma(x)/x^{-\gamma} \in (0, \infty)$ for each σ , then (5.2) holds with $L(\lambda) = \lambda^{\gamma/(\gamma+1)}$.
- (5) If there exists σ_1 and σ_2 such that $\log F_{\sigma_1}$ and $\log F_{\sigma_2}$ vary regularly as $x \rightarrow 0$ with different indices, then (5.2) fails to hold.

These are well-known facts in the Tauberian theory, see [1]. \square

For given positive i.i.d. random variables $\{\xi(z)\}_{z \in \mathbb{Z}^d}$, we define the passage time of a nearest neighbor path $r = (r_0, r_1, \dots, r_n)$ as

$$T(r, \xi) := \sum_{i=0}^{n-1} \xi(r_i),$$

where the right hand side is set to be 0 if $n = 0$. The travel time from x to y is defined as

$$T(x, y, \xi) := \inf\{T(r, \xi); r \text{ is a path from } x \text{ to } y\}.$$

It is shown by Cox and Durrett [2] that there exists a deterministic norm ν_ξ such that

$$(5.2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} T(0, nx, \xi) = \nu_\xi(x) \text{ in probability}$$

for all $x \in \mathbb{Z}^d$.

Proposition 5.2 For any $x \in \mathbb{R}^d$,

$$(5.3) \quad \frac{\alpha_\lambda(x)}{L(\lambda)} \rightarrow \nu_\Theta(x), \quad \lambda \rightarrow \infty.$$

Proof. It suffices show the assertion only for $x \in \mathbb{Z}^d$. Indeed, since both α_λ and ν_Θ are homogeneous, it extends to \mathbb{Q}^d and then to \mathbb{R}^d by continuity. Now for $x \in \mathbb{Z}^d$, we estimate the difference as

$$(5.4) \quad \begin{aligned} \left| \frac{\alpha_\lambda(x)}{L(\lambda)} - \nu_\Theta(x) \right| &\leq \left| \frac{\alpha_\lambda(x)}{L(\lambda)} - \frac{a_\lambda(0, nx, \omega, \sigma)}{nL(\lambda)} \right| \\ &\quad + \left| \frac{a_\lambda(0, nx, \omega, \sigma)}{nL(\lambda)} - \frac{1}{n} T(0, nx, \theta_{\lambda, \sigma}/L(\lambda)) \right| \\ &\quad + \left| \frac{1}{n} T(0, nx, \theta_{\lambda, \sigma}/L(\lambda)) - \nu_{\theta_{\lambda, \sigma}/L(\lambda)}(x) \right| \\ &\quad + \left| \nu_{\theta_{\lambda, \sigma}/L(\lambda)}(x) - \nu_\Theta(x) \right|, \end{aligned}$$

where $n \in \mathbb{N}$. Note that for any fixed $\lambda > 0$, the first and second terms converge to 0 in probability as $n \rightarrow \infty$. We also know that the fourth term in (5.4) tends to 0 as $\lambda \rightarrow \infty$ due to our assumption (5.1) and the continuity of the time constant shown in Theorem 6.9 in [4].

The following lemma gives a control on the second term.

Lemma 5.3 For any $\epsilon > 0$, there exists $\Lambda > 0$ such that for all $\lambda \geq \Lambda$,

$$(5.5) \quad \limsup_{n \rightarrow \infty} \mathbb{P} \otimes \mathbf{P} \left(\left| \frac{a_\lambda(0, nx, \omega, \sigma)}{nL(\lambda)} - \frac{1}{n} T(0, nx, \theta_{\lambda, \sigma}/L(\lambda)) \right| > \epsilon \right) < \epsilon.$$

Proof. Since one of the bound

$$\begin{aligned} \frac{a_\lambda(0, nx, \omega, \sigma)}{L(\lambda)} &\geq -\frac{1}{L(\lambda)} \log E_\omega^0[\exp\{-\lambda T(0, nx, \sigma)\} \mathbb{1}_{\{H^X(nx) < \infty\}}] \\ &\geq T(0, nx, \theta_{\lambda, \sigma}/L(\lambda)) \end{aligned}$$

is trivial, we have only to show that for any $\epsilon > 0$,

$$\mathbb{P} \otimes \mathbf{P} \left(\frac{a_\lambda(0, nx, \omega, \sigma)}{nL(\lambda)} \leq \frac{1}{n} T(0, nx, \theta_{\lambda, \sigma}/L(\lambda)) + \epsilon \right) > 1 - \epsilon$$

when λ and n are sufficiently large. To this end, we first pick a path $r = \{r_m\}_{m=0}^{N(r)}$ from those paths connecting 0 and nx and satisfying

$$(5.6) \quad T(r, \theta_{\lambda, \sigma}/L(\lambda)) \leq T(0, nx, \theta_{\lambda, \sigma}/L(\lambda)) + 1$$

by some deterministic rule.

Lemma 5.4 For any $x \in \mathbb{Z}^d$, there exists a constant $c_x > 0$ such that

$$(5.7) \quad \lim_{n \rightarrow \infty} \mathbf{P}(N(r) \leq c_x n) = 1,$$

where $N(r)$ is the length of the path r picked above.

Proof. Note first that

$$\lim_{n \rightarrow \infty} \mathbf{P}(T(0, nx, \theta_{\lambda, \sigma}/L(\lambda)) \leq 2\nu_\Theta(x)n) = 1$$

by (5.2) and the continuity of the time constant. Hence

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbf{P}(N(r) > cn) &\leq \limsup_{n \rightarrow \infty} \mathbf{P}(T(r, \theta_{\lambda, \sigma}/L(\lambda)) \leq 2\nu_\Theta(x)n) \\ (5.8) \quad &\leq \limsup_{n \rightarrow \infty} \exp \left\{ 2\nu_\Theta(x)n + \sum_{m=1}^{cn} \log \mathbf{E}[e^{-\theta_{\lambda, \sigma}(r_m)/L(\lambda)}] \right\}. \end{aligned}$$

This right-hand side is 0 if $c > -2\nu_\Theta(x) \inf_{\lambda \geq 0} \log \mathbf{E}[e^{-\theta_{\lambda, \sigma}(0)/L(\lambda)}]$. \square

By using the above path $r = (0 = r_0, r_1, \dots, r_{N(r)} = nx)$,

$$\begin{aligned} &\frac{a_\lambda(0, nx, \omega, \sigma)}{nL(\lambda)} \\ &\leq -\frac{1}{nL(\lambda)} \log E_\omega^x \left[\exp \left\{ -\lambda \sum_{m=0}^{H^X(y)-1} \sigma(X_m) \right\} \mathbb{1}_{\{(X_m)_{m=0}^{N(r)} = r\}} \right] \end{aligned}$$

$$\leq \frac{1}{n}T(0, nx, \theta_{\lambda, \sigma}/L(\lambda)) + \frac{1}{n} + \frac{1}{L(\lambda)} \sum_{m=0}^{N(r)-1} \frac{-\log \omega(r_m, r_{m+1} - r_m)}{n}.$$

Since the last sum is bounded with high probability, that is,

$$\lim_{n \rightarrow \infty} \mathbf{P} \left(\sum_{m=0}^{N(r)-1} \frac{-\log \omega(r_m, r_{m+1} - r_m)}{n} \leq 2c_x \mathbb{E}[-\log \max_{|e|=1} \omega(0, e)] \right) = 1$$

by Lemma 5.4 and the weak law of large numbers, we obtain the desired conclusion. \square

To complete the proof of Proposition 5.2, pick an arbitrary $\epsilon > 0$ and take $\lambda > 0$ so large that $|\nu_{\theta_{\lambda, \sigma}/L(\lambda)}(x) - \nu_{\Theta}(x)| < \epsilon$ and Lemma 5.3 hold. Then we know that the events

$$\begin{aligned} & \left\{ (\omega, \sigma) : \left| \frac{\alpha_{\lambda}(x)}{L(\lambda)} - \frac{a_{\lambda}(0, nx, \omega, \sigma)}{nL(\lambda)} \right| < \epsilon \right\}, \\ & \left\{ (\omega, \sigma) : \left| \frac{a_{\lambda}(0, nx, \omega, \sigma)}{nL(\lambda)} - \frac{1}{n}T(0, nx, \theta_{\lambda, \sigma}/L(\lambda)) \right| < \epsilon \right\}, \\ & \left\{ \sigma : \left| \frac{1}{n}T(0, nx, \theta_{\lambda, \sigma}/L(\lambda)) - \nu_{\theta_{\lambda, \sigma}/L(\lambda)}(x) \right| < \epsilon \right\} \end{aligned}$$

have probability tending to 1 as $n \rightarrow \infty$. In particular, we can find (ω, σ) belonging to all the events above and substituting it into (5.4), we obtain

$$\left| \frac{\alpha_{\lambda}(x)}{L(\lambda)} - \nu_{\Theta}(x) \right| < 4\epsilon.$$

\square

6. ASYMPTOTICS OF THE RATE FUNCTION

In this section, we discuss asymptotics of the rate function as $x \rightarrow \infty$ and $x \rightarrow 0$ in some special cases.

We start with the case $x \rightarrow \infty$. Let

$$\lambda^*(x) = \inf \left\{ \lambda \geq 0 : L(\lambda)\nu_{\Theta}(x) - \lambda = \sup_{\lambda \geq 0} (L(\lambda)\nu_{\Theta}(x) - \lambda) \right\}$$

with the convention $\inf \emptyset = \infty$.

Proposition 6.1 Suppose that the same assumption as in Proposition 5.2 holds. In addition, assume that $\lambda^*(x) < \infty$ for any $x \in \mathbb{R}^d$ and

$$(6.1) \quad \lim_{\ell \rightarrow \infty} L(\lambda^*(\ell x))\nu_{\Theta}(\ell x) / \lambda^*(\ell x) > 1.$$

Then for any $x \in \mathbb{R}^d$,

$$(6.2) \quad I(\ell x) = \sup_{\lambda \geq 0} (L(\lambda)\ell\nu_{\Theta}(x) - \lambda)(1 + o(1))$$

as $\ell \rightarrow \infty$.

Proof. Note that $\lambda^*(\ell x) \rightarrow \infty$ as $\ell \rightarrow \infty$. On the other hand, we know from Proposition 5.2 that $\alpha_\lambda(x) = L(\lambda)\nu(x)(1 + o(1))$ for some $\delta(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. Combining these two facts and using (6.1), we obtain

$$\begin{aligned} I(\ell x) &\geq \alpha_{\lambda^*(\ell x)}(\ell x) - \lambda^*(\ell x) \\ &= L(\lambda^*(\ell x))\ell\nu_\Theta(x)(1 + o(1)) - \lambda^*(\ell x) \\ &= (L(\lambda^*(\ell x))\ell\nu_\Theta(x) - \lambda^*(\ell x))(1 + o(1)) \end{aligned}$$

as $\ell \rightarrow \infty$. This proves the lower bound in (6.2). To prove the upper bound, fix any $\epsilon > 0$. Then by the same reasoning as above, for sufficiently large ℓ , we have

$$\sup_{\lambda \geq 0}(\alpha_\lambda(\ell x) - \lambda) \leq \sup_{\lambda \geq 0}((1 + \epsilon)L(\lambda)\ell\nu_\Theta(x) - \lambda)$$

and the right hand side is bounded from above by $(1 + 2\epsilon)\sup_{\lambda \geq 0}(\alpha_\lambda(\ell x) - \lambda)$. \square

Using the above proposition, one can see that in the situation of Example 5.1-(2),

$$I(\ell x) \sim \ell\nu_\Theta(x)(\log(\ell\nu_\Theta(x)) - 1) \text{ as } \ell \rightarrow \infty$$

and in that of Example 5.1-(4),

$$I(\ell x) \sim \frac{1}{1 + \gamma} \left(\frac{\gamma}{1 + \gamma} \right)^\gamma (\ell\nu_\Theta(x))^{1 + \gamma} \text{ as } \ell \rightarrow \infty.$$

Example 5.1-(1) does not fall within the scope of the above proposition but it is easy to see that $I(x) = \infty$ as soon as $\nu_\Theta(x) > 1$.

Let us turn to the case $x \rightarrow 0$. We only consider the *Simple Random Walk with Random Holding Times*, i.e. $\omega(x, e) = \frac{1}{2d}$ for all $x \in \mathbb{Z}^d$ and $|e| = 1$. This is of course very restrictive but it seems rather not reasonable to expect a unified result under the general setting as the large deviation of RWRE exhibits rich phenomena. For example, if a nestling RWRE satisfies the law of large numbers with nonzero speed v , then the rate function is zero on the line segment connecting the origin and v .

Proposition 6.2 Assume $\omega(x, e) = \frac{1}{2d}$ for all $x \in \mathbb{Z}^d$ and $|e| = 1$ for \mathbb{P} almost every ω . Then

$$(6.3) \quad I(\ell x) = \frac{d}{2} \mathbf{E} \left[\int_0^\infty s \sigma_0(ds) \right] \ell^2 |x|^2 (1 + o(1)) \text{ as } \ell \rightarrow 0.$$

Proof. Our α_λ is nothing but the quenched Lyapunov exponent of Green's function with the random potential $\theta_{\sigma, \lambda}$ (cf. [9]). It follows from our assumption $\int_0^\infty s \sigma_0(ds) \in L^d(\mathbf{P})$ that

$$\lambda^{-1} \theta_{\sigma, \lambda}(0) \rightarrow \int_0^\infty s \sigma_0(ds) \text{ as } \lambda \rightarrow 0$$

\mathbf{P} -a.s. and in $L^1(\mathbf{P})$. This verifies the assumption in Theorem 4 of [5], which tells us that

$$(6.4) \quad \alpha_\lambda(x) = \sqrt{2d\lambda \mathbf{E} \left[\int_0^\infty s \sigma_0(ds) \right]} |x| (1 + o(1)) \text{ as } \lambda \rightarrow 0.$$

From this asymptotics, one can deduce (6.3) by the same way as for Proposition 6.1. \square

REFERENCES

- [1] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, Vol. 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1989.
- [2] J. Theodore Cox and Richard Durrett. Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.*, Vol. 9, No. 4, pp. 583–603, 1981.
- [3] Amir Dembo, Nina Gantert, and Ofer Zeitouni. Large deviations for random walk in random environment with holding times. *Ann. Probab.*, Vol. 32, No. 1B, pp. 996–1029, 2004.
- [4] Harry Kesten. Aspects of first passage percolation. In *École d’été de probabilités de Saint-Flour, XIV—1984*, Vol. 1180 of *Lecture Notes in Math.*, pp. 125–264. Springer, Berlin, 1986.
- [5] Elena Kosygina, Thomas Mountford, and Martin Zerner. Lyapunov exponents of greenfs functions for random potentials tending to zero. *Probab. Theory Related Fields*, Vol. 150, pp. 43–59, 2011.
- [6] Firas Rassoul-Agha and Timo Seppäläinen. Process-level quenched large deviations for random walk in random environment. *Ann. Inst. Henri Poincaré Probab. Stat.*, Vol. 47, No. 1, pp. 214–242, 2011.
- [7] Alain-Sol Sznitman. Shape theorem, Lyapounov exponents, and large deviations for Brownian motion in a Poissonian potential. *Comm. Pure Appl. Math.*, Vol. 47, No. 12, pp. 1655–1688, 1994.
- [8] S. R. S. Varadhan. Large deviations for random walks in a random environment. *Comm. Pure Appl. Math.*, Vol. 56, No. 8, pp. 1222–1245, 2003. Dedicated to the memory of Jürgen K. Moser.
- [9] Martin P. W. Zerner. Directional decay of the Green’s function for a random nonnegative potential on \mathbf{Z}^d . *Ann. Appl. Probab.*, Vol. 8, No. 1, pp. 246–280, 1998.
- [10] Martin P. W. Zerner. Lyapounov exponents and quenched large deviations for multidimensional random walk in random environment. *Ann. Probab.*, Vol. 26, No. 4, pp. 1446–1476, 1998.

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